Approximating the Amplitude and Form of Limit Cycles in the Weakly Nonlinear Regime of Liénard Systems

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Abstract

Liénard equations, $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$, with $f(x)$ an even continuous function are considered. In the weakly nonlinear regime ($\epsilon \to 0$), the number and a $\mathcal{O}(\epsilon^0)$ approximation of the amplitude of limit cycles present in this type of systems, can be obtained by applying a methodology recently proposed by the authors [López-Ruiz R, López JL. Bifurcation curves of limit cycles in some Liénard systems. Int J Bifurcat Chaos 2000; 10:971-980]. In the present work, that method is carried forward to higher orders in ϵ and is embedded in a general recursive algorithm capable to approximate the form of the limit cycles and to correct their amplitudes as an expansion in powers of ϵ. Several examples showing the application of this scheme are given.

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1 Introduction

Self-sustained periodic oscillations are common in nature [1]. They are also important in engineering applications [2]. The calculation of the number and amplitude of such different periodic motions (limit cycles) taking place in an oscillating system is an unsolved problem. This question constitutes the second part of Hilbert's Sixteenth Problem [3, 4] when we are restricted to two-dimensional autonomous systems of the form:

$$\begin{aligned} \dot{x} &= P_n(x,y), \\ \dot{y} &= Q_n(x,y), \end{aligned} \tag{1}$$

where $\dot{x}(t) = dx(t)/dt$, $\dot{y}(t) = dy(t)/dt$ and (P_n, Q_n) are polynomials of degree n with real coefficients. Although it has been proved that the number of limit cycles in systems of type (1) is finite [5, 6], the determination of the maximal number $H(n)$ of such solutions for a given degree n is far away of being known. Even for $n = 2$, $H(2)$ is still not determined [7]. It has been verified with different examples, for instance, that $H(2) \geq 4$, $H(3) \geq 11$, $H(4) \geq 15$ and $H(5) \geq 23$ but the exact values of $H(n)$ for these cases [7, 8, 9, 10] are unknown.

The van der Pol oscillator $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$, is an example of system (1) that has been well studied. In this case, $P_3(x,y) = y$ and $Q_3(x,y) = -\epsilon(x^2-1)y - x$. It displays a limit cycle whose uniqueness and non-algebraicity has been shown for the whole range of the parameter ϵ [11]. Its behaviour runs from near-harmonic oscillations for ϵ close to zero ($\epsilon \to 0$) to relaxation oscillations when ϵ tends to infinity ($\epsilon \to \infty$), making it a good model for many practical situations [12].

A generalization of the van der Pol oscillator is the Liénard equation,

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0, \tag{2}$$

with ϵ a real parameter and $f(x)$ any real function. When $f(x)$ is a polynomial of degree $N = 2n+1$ or $2n$ this equation takes the form (1) with $P_{N+1}(x,y) = y$ and $Q_{N+1}(x,y) = -\epsilon f(x)y - x$. It has been conjectured by Lins, Melo and Pugh (LMP-conjecture) that the

maximum number of limit cycles allowed is just n [13]. It is true if $N = 2$, or $N = 3$ or if $f(x)$ is even and $N = 4$ [13, 14]. Also, there are strong arguments for claiming its truth in the strongly nonlinear regime ($\epsilon \to \infty$) when $f(x)$ is an even polynomial [15]. There are no general results about the limit cycles when $f(x)$ is a polynomial of degree greater than 5 neither, in general, when $f(x)$ is an arbitrary real function [16].

The calculation of the form of limit cycles is another difficult task associated to this problem. As far as we know, there is no a general methodology documented in the literature on this subject. The different schemes used to calculate the number and amplitude of the periodic motions in the weak nonlinear regime, namely the averaging and perturbation methods [2, 17], could also be applied to find the approximate wave form of those orbits in the time or frequency domain [18]. In general, the insistence in performing these calculations in phase space with the time variable being explicit can complicate the achievement of the objective.

In this work, we undertake the task of calculating the form of limit cycles in phase space for the weakly nonlinear regime of Liénard equations. The method exploited in [19, 20] to find the number and a first order approximation to the amplitude of limit cycles when $| \epsilon | \ll 1$ is recalled in Section 2. The embedding of this method in a general recursive framework for calculating the form and amplitude of those periodic solutions is presented in Section 3. The general algorithm is rewritten in Section 4. Some illustrative examples are given in Section 5. Finally, we present our conclusions.

2 Integral equation for the limit cycles

In order to study the limit cycles of equation (2) with the time variable being implicit, it is convenient to rewrite it in the coordinates $(x, \dot{x}) = (x, y)$ in the plane, with $\dot{x}(t) = y(x)$ and $\ddot{x}(t) = y(x)y'(x)$ (where $y'(x) = dy/dx$):

$$yy' + \epsilon f(x)y + x = 0. \tag{3}$$

A limit cycle $C_l \equiv (x, y_\pm(x))$ of equation (3) has a positive branch $y_+(x) > 0$ and a negative branch $y_-(x) < 0$. They cut the x-axis in two points $(-a_-, 0)$ and $(a_+, 0)$ with $a_-, a_+ > 0$ because the origin $(0,0)$ is the only fixed point of Eq. (3). Then every limit cycle C_l solution of Eq. (3) encloses the origin and the oscillation x runs in the interval $-a_- < x < a_+$. The amplitudes of oscillation a_-, a_+ identify the limit cycle. The result is a nested set of closed curves that defines the qualitative distribution of the integral curves in the plane (x, y). The stability of the limit cycles is alternate. For a given stable limit cycle, the two neighbouring limit cycles, the closest one in its interior and the closest one in its exterior, are unstable, and viceversa.

When $f(x)$ is an even function, the *inversion symmetry* $(x, y) \leftrightarrow (-x, -y)$ of Eq. (3) implies $y_+(x) = -y_-(-x)$ and then $a_1 = a_2 = a$. Therefore we can restrict ourselves to the positive branches of the limit cycles $(x, y_+(x))$ with $-a \leq x \leq a$. In this case, the amplitude a identifies the limit cycle. The *parameter inversion symmetry* $(\epsilon, x, y) \leftrightarrow (-\epsilon, x, -y)$ implies that if $C_l \equiv (x, y_\pm(x))$ is a limit cycle for a given ϵ, then $\overline{C}_l \equiv (x, -y_\mp(x))$ is a limit cycle for $-\epsilon$. In consequence, the amplitude a of the limit cycles in these Liénard systems is an even function of ϵ. Moreover if C_l is stable (or unstable) then $\overline{C}_l$ is unstable (or stable, respectively). Therefore it is enough to consider the limit cycles when $\epsilon > 0$ for obtaining all the periodic solutions. (The limit cycles for a given $\epsilon < 0$ are obtained from a reflection over the x-axis of those limit cycles obtained for $\epsilon > 0$).

Another global property of a limit cycle of Eq. (3) can be derived from the fact that the mechanical energy $E = (x^2 + y^2)/2$ is conserved in a half oscillation:

$$\int_{-a}^{a} \frac{dE}{dx} dx = 0. \tag{4}$$

By Eq. (3), we have $\frac{dE}{dx} = -\epsilon f(x) y$ and, hence, by substituting it in the last expression, an integral equation is obtained for the limit cycle, that is:

$$\int_{-a}^{a} f(x) y_+(x) dx = 0. \tag{5}$$

The finite set of limit cycles of Eq. (3) also verifies Eq. (5). If the limit cycles are expanded in a power series of ϵ, Eq. (3) imposes the differential relationships that must be verified

for the different orders in ϵ; this is exploited in Sections 3.2 and 3.3. An alternative point of view comes from the necessary integral condition (5). It can also be exploited to find the correct approximate amplitudes of the limit cycles up to a given order in ϵ; this is carried out in Section 3.1 up to the order $\mathcal{O}(\epsilon^0)$ and in Section 4 up to any order.

3 Amplitude and form of the limit cycles

We proceed now to explain how to calculate recursively the form and amplitude of limit cycles of Eq. (3) for different orders in ϵ when this parameter is small and $f(x)$ is even.

3.1 Limit cycles at order zero

For a given $y(x)$ we define the function $S(a)$ as follows,

$$S(a) \equiv \int_{-a}^{a} f(x)y(x)dx. \tag{6}$$

Then, as it has been established in Eq. (5) of the previous section, a necessary condition for $y_+(x)$, hereinafter called $y(x)$, to be a limit cycle of Eq. (3) with amplitude a is

$$S(a) = 0. \tag{7}$$

When $\epsilon \to 0$, the limit cycle solutions of Eq. (3) emerging from the period annulus surrounding the center $(0,0)$ become circles on the plane,

$$y_0(x) = \sqrt{a^2 - x^2}. \tag{8}$$

The amplitudes of these limit cycles, a, verify $S(a) = 0$:

$$\beta_0(a) \equiv \int_{-a}^{a} f(x)y_0(x)dx = \int_{-a}^{a} f(x)\sqrt{a^2 - x^2}dx = 0, \tag{9}$$

and every solution a of $\beta_0(a) = 0$ is the germ of at least one limit cycle of amplitude a [19]. This is a well established result if we recall at this point [4] that $\beta_0(a)$ coincides with the first Melnikov function or, equivalently, with the Abelian integral defined for the

perturbed Hamiltonian system (2) whose level curves at $\epsilon = 0$ are given by $x^2 + y^2 = a^2$. For instance, when $f(x)$ is an even polynomial of degree $2n$, $\beta_0(a)$ is a^2 times a polynomial of degree n in a^2. Then, it has at most n simple positive roots, and we can conclude that LMP-conjecture is true in this regime [19].

3.2 Limit cycles up to order $\mathcal{O}(\epsilon^N)$

By continuity, we can assume that the limit cycle $y(x)$ for small ϵ is approximated by $y_0(x)$ plus an expansion in powers of ϵ:

$$y(x) = y_0(x) + \sum_{n=1}^{N} \epsilon^n y_n(x) + \mathcal{O}(\epsilon^{N+1}). \tag{10}$$

Replacing this expansion in (3) and equating powers of ϵ, every function $y_n(x)$ of the expansion satisfies a first order linear differential equation of the form:

$$(y_0 y_n)' + W_{n-1} = 0, \qquad n = 1, 2, 3, ..., N, \tag{11}$$

where

$$W_{n-1}(x) \equiv \begin{cases} y_0(x) f(x) & \text{if } n = 1, \\ \sum_{k=1}^{n-1} y_k'(x) y_{n-k}(x) + y_{n-1}(x) f(x) & \text{if } 2 \le n \le N. \end{cases} \tag{12}$$

The system of equations (11) can be iteratively solved for $y_n(x)$ by imposing the contour condition $y_n(-a) = 0 \; \forall \; n$. Then, we obtain:

$$\begin{aligned} y_n(x) &= -\frac{1}{y_0(x)} \int_{-a}^{x} W_{n-1}(t) dt = \\ &= -\frac{1}{y_0(x)} \left\{ \frac{1}{2} \sum_{k=1}^{n-1} y_k(x) y_{n-k}(x) + \int_{-a}^{x} y_{n-1}(t) f(t) dt \right\}. \end{aligned} \tag{13}$$

Therefore, for small ϵ, the solutions $y(x)$ of Eq. (3), which verify $y(-a) = 0$, are approximated up to the order $\mathcal{O}(\epsilon^N)$ by

$$y(x) \simeq y^{(N)}(x) \equiv \sum_{n=0}^{N} \epsilon^n y_n(x). \tag{14}$$

For example, up to the order $\mathcal{O}(\epsilon^3)$, $y(x)$ can be written as

$$y^{(3)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \epsilon^3 y_3(x), \tag{15}$$

where

$$y_1(x) = -\frac{1}{y_0(x)}\int_{-a}^{x} y_0(t)f(t)dt, \tag{16}$$

$$y_2(x) = -\frac{y_1^2(x)}{2y_0(x)} - \frac{1}{y_0(x)}\int_{-a}^{x} y_1(t)f(t)dt, \tag{17}$$

$$y_3(x) = -\frac{y_1(x)y_2(x)}{y_0(x)} - \frac{1}{y_0(x)}\int_{-a}^{x} y_2(t)f(t)dt. \tag{18}$$

The possible amplitudes a of the limit cycles of Eq. (3), up to the order $\mathcal{O}(\epsilon^N)$, are determined by imposing that the solution $y^{(N)}(x)$ vanishes also at $x = a$:

$$y^{(N)}(a) = \sum_{n=0}^{N} \epsilon^n y_n(a) = 0. \tag{19}$$

The condition $y^{(N)}(a) = 0$ identifies the limit cycle $y(x)$ up to the order $\mathcal{O}(\epsilon^N)$, but its amplitude a only up to the order $\mathcal{O}(\epsilon^{N-1})$ because $y_0(a)$ vanishes for any value of a. Hence, the last condition (19) can be rewritten as

$$y^{(N)}(a) = 0 \quad \Longrightarrow \quad \sum_{n=1}^{N} \epsilon^{n-1} y_n(a) = 0. \tag{20}$$

The solutions a from this last equation are used to calculate $y_N(x)$, and hence the approximation $y^{(N)}(x)$ for the form of the limit cycles can be finally obtained. The iteration of this scheme for $N = 1, 2, 3\ldots$ generates the whole expansion of the limit cycle in powers of ϵ.

3.3 Application of the method for the lower orders

3.3.1 Limit cycles up to the order $\mathcal{O}(\epsilon^1)$

Up to the order $\mathcal{O}(1)$ in the amplitude ($N = 1$), the above condition (20) imposes the following restriction over the values of a:

$$y^{(1)}(a) = 0 \quad \Longrightarrow \quad y_1(a) = 0. \tag{21}$$

The function $y_1(x)$ reads

$$y_1(x) = -\frac{1}{y_0(x)}\int_{-a}^{x} y_0(t)f(t)dt = -\frac{1}{y_0(x)}\left\{\beta_0(a) + \int_{a}^{x} y_0(t)f(t)dt\right\}, \tag{22}$$

where $\beta_0(a)$ is given in expression (9). The function $y_0(t)f(t)$ is continuous in the interval $[-a, a]$ and of the order $\mathcal{O}(\sqrt{a-x})$ when $x \to a$. Therefore,

$$\int_a^x y_0(t)f(t)dt = \mathcal{O}\left((a-x)^{3/2}\right) \qquad \text{when } \; x \to a. \tag{23}$$

As $y_0(x) = \sqrt{a^2 - x^2}$, then, for any $a > 0$,

$$y_1(x) = -\frac{\beta_0(a)}{y_0(x)} + \mathcal{O}(a-x), \qquad \text{when } \; x \to a. \tag{24}$$

Hence, from (22), a necessary and sufficient condition for $y^{(1)}(x)$ to be zero at $x = a$ is just the condition (9) derived from the integral equation (5) for the zero order approximation:

$$\beta_0(a) = 0. \tag{25}$$

We recover in this way the order $\mathcal{O}(\epsilon^0)$ condition for the amplitudes of the limit cycles emerging from the period annulus at order zero (9). The form of these solutions is perturbed at the order $\mathcal{O}(\epsilon^1)$ and reads:

$$y^{(1)}(x) = y_0(x) + \epsilon y_1(x), \tag{26}$$

with $y_1(x)$ given by expression (16) when a is solution of Eq. (25).

3.3.2 Limit cycles up to the order $\mathcal{O}(\epsilon^2)$

Up to order $\mathcal{O}(\epsilon)$ in the amplitude ($N = 2$), condition (20) imposes the following restriction over the values of a:

$$y^{(2)}(a) = 0 \quad \Longrightarrow \quad y_1(a) + \epsilon y_2(a) = 0. \tag{27}$$

Define

$$\beta_1(a) \equiv \int_{-a}^{a} y_1(x)f(x)dx. \tag{28}$$

Recalling that $f(x)y_0(x)$ is an even function of x, and then $\beta_0(a) = 2\int_0^a f(x)y_0(x)dx = 2\int_{-a}^0 f(x)y_0(x)dx$, we have that:

$$\beta_1(a) = -\int_{-a}^{a} \frac{f(x)}{y_0(x)}dx \int_{-a}^{x} f(t)y_0(t)dt = -\int_{-a}^{a} \frac{f(x)}{y_0(x)}dx \left\{\frac{1}{2}\beta_0(a) + \int_0^x f(t)y_0(t)dt\right\}.$$

Because $f(t)y_0(t)$ is an even function of t, the $t-$integral in the above last expression is an odd function of x and then vanishes with the exterior integral in x, when it runs in the interval $[-a, a]$. The result is

$$\beta_1(a) = -\frac{1}{2}\beta_0(a)\int_{-a}^{a}\frac{f(x)}{y_0(x)}dx = -\frac{1}{2a}\beta_0(a)\beta_0'(a). \tag{29}$$

From this equation and (17) we find out that

$$y_2(x) = -\frac{1}{y_0(x)}\left\{\frac{1}{2}y_1^2(x) - \frac{1}{2a}\beta_0(a)\beta_0'(a) + \int_a^x y_1(t)f(t)dt\right\}. \tag{30}$$

Observe that if a is a solution of $\beta_0(a) = 0$, that is $y_1(a) = 0$, the same value of a automatically satisfies $y_2(a) = 0$. This means that, up to the order $\mathcal{O}(\epsilon)$, the amplitudes of the limit cycles of Eq. (3) are the positive roots of the polynomial $\beta_0(a)$. Then, it seems that they do not experience any correction at the order $\mathcal{O}(\epsilon)$. Let's show this statement more precisely. Write $a = a_0 + a_1\epsilon + \mathcal{O}(\epsilon^2)$ when $\epsilon \longrightarrow 0$, where a_0 is a root of $\beta_0(a)$: $\beta_0(a_0) = 0$ and a_1 is a real number. Then,

$$\beta_0(a) = \beta_0'(a_0)a_1\epsilon + \mathcal{O}(\epsilon^2). \tag{31}$$

In the foregoing discussion we will consider that the limit $\epsilon \longrightarrow 0$ must be taken before the limit $x \longrightarrow a$ and then any product of symbols $\mathcal{O}(\epsilon^p)$ and $\mathcal{O}\left((a-x)^q\right)$ can be replaced by $\mathcal{O}(\epsilon^p)$. For any $a > 0$, $y_1(x)$ is a continuous functions in $[-a, a]$ up to the order $\mathcal{O}(\epsilon)$ and from (24) and (31):

$$y_1(x) = \frac{-\beta_0(a)}{y_0(x)} + \mathcal{O}(a-x) = \mathcal{O}(\epsilon) + \mathcal{O}(a-x). \tag{32}$$

Then,

$$\int_a^x y_1(t)f(t)dt = \mathcal{O}\left((a-x)^2\right) + \mathcal{O}(\epsilon) \qquad \text{when } x \longrightarrow a \quad \text{and } \epsilon \longrightarrow 0 \tag{33}$$

From the above equation and (30):

$$y_2(x) = -\frac{1}{y_0(x)}\left[\mathcal{O}\left((a-x)^2\right) + \beta_1(a) + \mathcal{O}(\epsilon)\right]. \tag{34}$$

From (24) and (34) we have that:

$$y^{(2)}(x) = y_0(x) + \epsilon\left[\mathcal{O}(a-x) - \frac{\beta_0(a)}{y_0(x)}\right] + \epsilon^2\left[\mathcal{O}\left((a-x)^{3/2}\right) - \frac{\beta_1(a)}{y_0(x)}\right] + \mathcal{O}(\epsilon^3).$$

Hence, in order to have $y^{(2)}(a) = 0$ up to the order $\mathcal{O}(\epsilon)$, the function

$$\beta^{(1)}(a, \epsilon) \equiv \beta_0(a) + \epsilon\beta_1(a) \tag{35}$$

must satisfy

$$\beta^{(1)}(a, \epsilon) = \mathcal{O}(\epsilon^2). \tag{36}$$

Using that $\beta_0(a) = \beta_0'(a_0)a_1\epsilon + \mathcal{O}(\epsilon^2)$ and $\beta_1(a) = \beta_1(a_0) + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon)$ (see (29)), the above equation implies

$$\beta_0'(a_0)a_1 = 0. \tag{37}$$

Therefore $a_1 = 0$ when $\beta_0'(a_0) \neq 0$. The case $\beta_0'(a_0) = 0$, where a bifurcation of a multiple limit cycle is possible, is not considered here. This means that the amplitude of the limit cycles of Eq. (3) have the form $a(\epsilon) = a_0 + a_2\epsilon^2 + \mathcal{O}(\epsilon^3)$, where a_2 is a real number. This is a consequence of the parameter inversion symmetry of Eq. (3) commented in Section 2.

The form of these solutions is perturbed and reads:

$$y^{(2)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x), \tag{38}$$

where $y_1(x), y_2(x)$ are given by expressions (16-17), with $y_2(x)$ calculated for the solutions a of $\beta_0(a) = 0$. Observe that as the function $y_0(x)$ is an even function of x, then, from (22), we see that $y_1(x)$ is an odd function of x when a is a root of $\beta_0(a)$, and then, from (17) and (29), it is clear that $y_2(x)$ is an even function of x.

3.3.3 Limit cycles up to the order $\mathcal{O}(\epsilon^3)$

Up to order $\mathcal{O}(\epsilon^2)$ in the amplitude ($N = 3$), condition (20) imposes the following restriction over the values of a:

$$y^{(3)}(a) = 0 \quad \Longrightarrow \quad y_1(a) + \epsilon y_2(a) + \epsilon^2 y_3(a) = 0. \tag{39}$$

From the definition (18) of $y_3(x)$ we have that

$$\begin{aligned} y_3(x) &= -\frac{1}{y_0(x)}\left[y_1(x)y_2(x) + \int_{-a}^{x} y_2(t)f(t)dt\right] = & (40)\\ &= -\frac{1}{y_0(x)}\left[y_1(x)y_2(x) + \int_{a}^{x} y_2(t)f(t)dt + \int_{-a}^{a} y_2(t)f(t)dt\right]. & (41)\end{aligned}$$

From the previous section we have that $\beta_0(a) = \beta_0'(a_0)a_2\epsilon^2 + \mathcal{O}(\epsilon^3)$ and $\beta_1(a) = \mathcal{O}(\epsilon^2)$. Then, from (24) and (30) we have that, for any $a > 0$ and up to the order $\mathcal{O}(\epsilon^2)$, $y_1(x)$ and $y_2(x)$ are continuous functions in $[-a, a]$, $y_1(x) = \mathcal{O}(a - x) + \mathcal{O}(\epsilon^2)$ and

$$y_2(x) = \mathcal{O}\left((a - x)^{3/2}\right) + \mathcal{O}(\epsilon^2). \tag{42}$$

Then:

$$\int_a^x y_2(t)f(t)dt = \mathcal{O}\left((a - x)^{5/2}\right) + \mathcal{O}(\epsilon^2) \qquad \text{when } x \to a \quad \text{and } \epsilon \to 0 \tag{43}$$

and

$$y_3(x) = -\frac{1}{y_0(x)}\left[\mathcal{O}\left((a - x)^{5/2}\right) + \beta_2(a) + \mathcal{O}(\epsilon^2)\right], \tag{44}$$

where we have defined

$$\beta_2(a) \equiv \int_{-a}^a y_2(t)f(t)dt. \tag{45}$$

If the results (24), (42) and (44) are substituted in the expression (15) we obtain:

$$\begin{aligned} y^{(3)}(x) &= y_0(x) + \epsilon\left[\mathcal{O}(a - x) - \frac{\beta_0(a)}{y_0(x)}\right] + \epsilon^2\left[\mathcal{O}\left((a - x)^{3/2}\right) + \mathcal{O}(\epsilon^2)\right] + \\ &\quad + \epsilon^3\left[\mathcal{O}\left((a - x)^2\right) - \frac{\beta_2(a)}{y_0(x)} + \mathcal{O}(\epsilon^2)\right] + \mathcal{O}(\epsilon^4). \end{aligned} \tag{46}$$

Using that $\beta_0(a) = \beta_0'(a_0)a_2\epsilon^2 + \mathcal{O}(\epsilon^3)$ and $\beta_2(a) = \beta_2(a_0) + \mathcal{O}(\epsilon)$ we have that

$$\begin{aligned} y^{(3)}(x) &= y_0(x) + \epsilon\left[\mathcal{O}(a - x) - \frac{\beta_0'(a_0)a_2}{y_0(x)}\epsilon^2\right] + \epsilon^2\mathcal{O}\left((a - x)^{3/2}\right) + \\ &\quad + \epsilon^3\left[\mathcal{O}\left((a - x)^2\right) - \frac{\beta_2(a_0)}{y_0(x)}\right] + \mathcal{O}(\epsilon^4). \end{aligned} \tag{47}$$

Hence, in order to have $y^{(3)}(a) = 0$ up to the order $\mathcal{O}(\epsilon^2)$, the function

$$\beta^{(2)}(a, \epsilon) \equiv \beta_0(a) + \epsilon\beta_1(a) + \epsilon^2\beta_2(a) = \left[\beta_0'(a_0)a_2 + \beta_2(a_0)\right]\epsilon^2 + \mathcal{O}(\epsilon^3)$$

must satisfy

$$\beta^{(2)}(a, \epsilon) = \mathcal{O}(\epsilon^3). \tag{48}$$

Therefore

$$a_2 = -\frac{\beta_2(a_0)}{\beta_0'(a_0)}. \tag{49}$$

The correction of the amplitude up to order $\mathcal{O}(\epsilon^2)$ is finally obtained:

$$a(\epsilon) = a_0 - \frac{\beta_2(a_0)}{\beta_0'(a_0)}\,\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{50}$$

and the form of the solutions at the order $\mathcal{O}(\epsilon^3)$ reads:

$$y^{(3)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \epsilon^3 y_3(x), \tag{51}$$

where $y_1(x), y_2(x), y_3(x)$ are given by expressions (16-18), with $y_3(x)$ calculated for the amplitudes $a(\epsilon)$ given by the expression (50).

4 An alternative view: the recursion of the integral equation

Let us see that the method explained in the last section can be viewed in an equivalent and alternative way as a recursive approximation in successive powers of ϵ to the integral Eq. (5).

4.1 General algorithm

Step 0: Set $y_0(x) = \sqrt{a^2 - x^2}$ and $N = 0$.

Step 1: Define the approximation of $y(x)$ at order $\mathcal{O}(\epsilon^N)$ given by Eq. (14):

$$y(x) \simeq y^{(N)}(x) \equiv \sum_{n=0}^{N} \epsilon^n y_n(x), \tag{52}$$

and find the solutions a of the integral equation at order $\mathcal{O}(\epsilon^N)$:

$$\beta^{(N)}(a,\epsilon) \equiv \int_{-a}^{a} f(x) y^{(N)}(x) dx = 0, \tag{53}$$

that is equivalent to the equation

$$\beta^{(N)}(a,\epsilon) \equiv \sum_{n=0}^{N} \epsilon^n \beta_n(a) = 0, \tag{54}$$

where

$$\beta_n(a) \equiv \int_{-a}^{a} f(x) y_n(x) dx = 0. \tag{55}$$

Step 2: For each solution a of *step 1* calculate $y_{N+1}(x)$ by the formula (13):

$$y_{N+1}(x) \quad = \quad -\frac{1}{y_0(x)} \int_{-a}^{x} W_N(t) dt =$$

$$= -\frac{1}{y_0(x)}\left\{\frac{1}{2}\sum_{k=1}^{N} y_k(x)y_{n-k}(x) + \int_{-a}^{x} y_N(t)f(t)dt\right\}. \qquad (56)$$

Step 3: Replace N by $N+1$ and come back to *step 1*.

Note: It can be easily found that the application of this algorithm for the lower orders, up to $N = 2$, repeats the results obtained in Section 3.3.

5 Examples

The limit cycles in the weak and in the strongly nonlinear regimes of different families of Liénard systems were studied in [19]. Here we perform the calculations proposed in Section 3 for two concrete examples, which are particular cases of the families 1 and 3 worked out in [19].

Example 1. The van der Pol oscillator is given for $f(x) = x^2 - 1$. This system has a unique limit cycle, which is stable for $\epsilon > 0$. Hence, the only root of $\beta_0(a)$ is $a_0 = 2$. For this value of a_0, we have $y^{(2)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x)$ with

$$y_0(x) = \sqrt{4 - x^2},$$

$$y_1(x) = \frac{x}{4}(4 - x^2)$$

and

$$y_2(x) = \frac{2 + x^2}{96}(4 - x^2)^{3/2}.$$

We integrate Eq. (3) by a Runge-Kutta method in order to obtain the limit cycle. This curve is plotted in a continuous trace in Fig. 1(a-b) for $\epsilon = 0.5$ and $\epsilon = 1$, respectively. The approximated limit cycle $y^{(2)}(x)$ is also plotted in those figures with a discontinuous trace for the same values of ϵ. Let us remark that, in this case, even up to $\epsilon = 3$, the approximation $y^{(2)}(x)$ to the limit cycle is very good.

The solution of $\beta^{(2)}(a, \epsilon) = 0$, up to order $\mathcal{O}(\epsilon^2)$, gives us:

$$\beta_2(a_0) = -\frac{\pi}{48}, \qquad \beta_0(a) = \frac{\pi}{8}a^2(a^2 - 4),$$

then

$$a(\epsilon) = 2 + \frac{1}{96}\,\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{57}$$

We must stress at this point that this expansion was also done for the van der Pol system in [21] by a perturbation method based on integrating factors. The author reported in that article [21] the expansion:

$$a(\epsilon) = 2 + \frac{23}{96}\,\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{58}$$

Our calculation (57) does not agree with this result, but agrees with the computational calculation of the 'exact' amplitudes given in Table 1.

Example 2. The same process is performed for $f(x) = 5x^4 - 9x^2 + 1$. In this case, the system has two limit cycles, one stable and the other unstable. The polynomial $\beta_0(a)$ has two positive roots: $a_0 = \sqrt{\frac{9-\sqrt{41}}{5}} = 0.720677$ (unstable limit cycle for $\epsilon > 0$) and $\bar{a}_0 = \sqrt{\frac{9+\sqrt{41}}{5}} = 1.755170$ (stable limit cycle for $\epsilon > 0$).

For the first limit cycle we have $y^{(2)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x)$ with

$$y_0(x) = \sqrt{a_0^2 - x^2},$$

$$y_1(x) = \frac{x}{24}(27 + 3\sqrt{41} - 20x^2)(x^2 - a_0^2)$$

and

$$y_2(x) = \frac{(a_0^2 - x^2)^{3/2}}{2880}\left[\frac{3222 - 218\sqrt{41}}{5} + (1003 + 63\sqrt{41})x^2 - 15(27 + 7\sqrt{41})x^4 + 200x^6\right].$$

The solution of $\beta^{(2)}(a,\epsilon) = 0$, up to order $\mathcal{O}(\epsilon^2)$, gives us:

$$\beta_2(a_0) = -0.007357, \qquad \beta_0(a) = \frac{\pi}{16}a^2(5a^4 - 18a^2 + 8).$$

Then

$$a(\epsilon) = 0.720677 + 0.003908\,\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{59}$$

For the second limit cycle we have $y^{(2)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x)$ with

$$y_0(x) = \sqrt{\bar{a}_0^2 - x^2},$$

$$y_1(x) = x\left(\frac{9-\sqrt{41}}{8} - \frac{5}{6}x^2\right)(x^2 - \bar{a}_0^2)$$

and

$$y_2(x) = \frac{(\bar{a}_0^2 - x^2)^{3/2}}{2880}\left[\frac{3222+218\sqrt{41}}{5} + (1003 - 63\sqrt{41})x^2 + 15(7\sqrt{41} - 27)x^4 + 200x^6\right].$$

The solution of $\beta^{(2)}(a,\epsilon) = 0$, up to order $\mathcal{O}(\epsilon^2)$, gives us:

$$\beta_2(\bar{a}_0) = -0.486199, \qquad \beta_0(a) = \frac{\pi}{16}a^2(5a^4 - 18a^2 + 8).$$

Then

$$a(\epsilon) = 1.755170 - 0.017880\ \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{60}$$

The comparison between the 'exact' and the approximated limit cycles can be seen in the plots of Fig. 2(a-b) and Fig. 3(a-b).

6 Conclusions

A general algorithm to approximate the form and the amplitude of limit cycles in the weakly nonlinear regime of Liénard equations has been presented. In each iteration of the method a new order in ϵ is calculated. Thus, the new term added in the expansion of ϵ for the form of the limit cycle must verify the differential equation, and the correction to this order for the amplitude of the limit cycle is derived from the integral equation. Different examples have been worked out and the results are in good agreement with the direct integration of the limit cycles.

In this paper we only have detailed the approximation up to the order ϵ^3. This process can be rewritten in a formalism that allow us to go farther in the order of approximation. We are just working in this direction and these new results will be presented elsewhere.

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References

[1] Winfree A. *The timing of biological clocks*. Scientific American Library, W.H. Freeman Co.; 1986.

[2] Andronov AA, Vitt AA, Khaikin SE. *Theory of Oscillators*, Dover, New York; 1989.

[3] Ilyashenko Y. Centennial history of Hilbert's 16th problem. Bull Amer Math Soc 2002; 39: 301-354.

[4] Li J. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. Int J Bif Chaos 2003; 13:47-106.

[5] Ecalle J, Martinet J, Mousse R, Ramis JP. Non-accumulation des cycles limites I-II. C R Acad Sci Paris I 1987; 304:375-431.

[6] Ilyashenko Y. Finitess theorems for limit cycles. Russian Math Surveys 1990; 45:129-203.

[7] Ye Y. *Theory of limit cycles*. In: Trans Math Monographs, vol. 66. Boston: American Mathematical Society; 1986.

[8] Li J, Huang Q. Bifurcations of limit cycles forming compound eyes in the cubic system. Chin Ann Math 1987; 8B:391-403.

[9] Zhang T, Han M, Zan H, Meng X. Bifurcations of limit cycles for a cubic Hamiltonian system under quartic perturbations. Chaos, Solitons & Fractals 2004; 22:1127-1138.

[10] Li J, Chang HSY, Chung KW. Bifurcations of limit cycles in a Z_6-equivariant planar vector field of degree 5. Sci China Ser A 2002; 45: 817-826.

[11] Odani K. The limit cycle of the van der Pol equation is not algebraic. J Differen Equat 1995; 115:146-152.

[12] López-Ruiz R, Pomeau Y. Transition between two oscillation modes. Phys Rev E 1997; 55: R3820-R3823.

[13] Lins A, de Melo W, Pugh CC. On Liénard's equation. In: Lectures Notes in Math, vol. 597. Springer-Verlag; 1977. p. 355.

[14] Rychkov GS. The maximum number of limit cycles of the system $\dot{y} = -x$, $\dot{x} = y - \sum_{i=0}^{2} a_i x^{2i+1}$ is two. Differen Equat 1975;11:301-302.

[15] López JL, López-Ruiz R. The limit cycles of Liénard equations in the strongly non-linear regime. Chaos, Solitons & Fractals 2000; 11:747-756.

[16] Giacomini H, Neukirch S. Number of limit cycles of the Liénard equation. Phys Rev E 1997; 56:3809-3813.

[17] Jordan DW, Smith P. *Nonlinear ordinary differential equations.* Oxford University Press, Oxford; 1977.

[18] Padín MS, Robbio FI, Moiola JL, Chen G. On limit cycle approximations in the van der Pol oscillator. Chaos, Solitons & Fractals 2005; 23:207-220.

[19] López-Ruiz R, López JL. Bifurcation curves of limit cycles in some Liénard systems. Int J Bifurcat Chaos 2000; 10:971-980.

[20] López JL, López-Ruiz R. Number and amplitude of limit cycles emerging from topologically equivalent perturbed centers. Chaos, Solitons & Fractals 2003; 17:135-143.

[21] Van Horssen WT. A perturbation method based on integrating factors. SIAM J. Appl. Math. 1999; 59:1427-1443.

Table 1. The value a_T represents the approximated amplitude $a(\epsilon)$ of the van der Pol limit cycle obtained from (57) for the indicated values of ϵ. The value a_E represents the amplitude a obtained by integrating directly the system with a Runge-Kutta method.

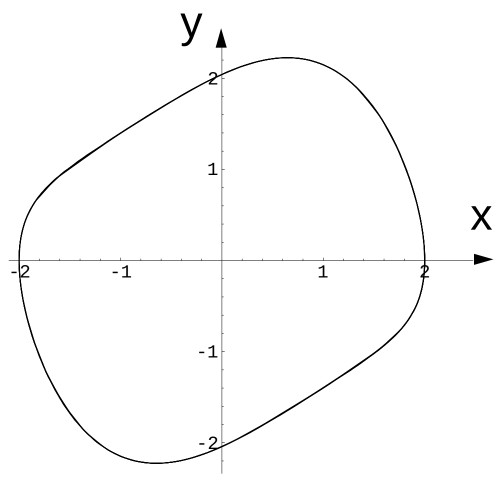
ϵ	0.1	0.2	0.3	0.4	0.5
a_T	2.00010	2.00041	2.00093	2.00166	2.00260
a_E	2.00010	2.00041	2.00092	2.00161	2.00248
ϵ	0.6	0.7	0.8	0.9	1
a_T	2.00375	2.00510	2.00666	2.00843	2.01041
a_E	2.00351	2.00466	2.00591	2.00724	2.00862

Figures

Fig. 1a-b: Exact limit cycle (continuous line) and approximated limit cycle (discontinuous line) up to order $\mathcal{O}(\epsilon^2)$ for the van der Pol system with (a) $\epsilon = 0.5$ and (b) $\epsilon = 1$.

Fig. 2a-b: Exact limit cycle (continuous line) and approximated limit cycle (discontinuous line) up to order $\mathcal{O}(\epsilon^2)$ for the smallest limit cycle of the example 2 with (a) $\epsilon = -0.5$ and (b) $\epsilon = -1$.

Fig. 3a-b: Exact limit cycle (continuous line) and approximated limit cycle (discontinuous line) up to order $\mathcal{O}(\epsilon^2)$ for the biggest limit cycle of the example 2 with (a) $\epsilon = 0.5$ and (b) $\epsilon = 1$.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-1
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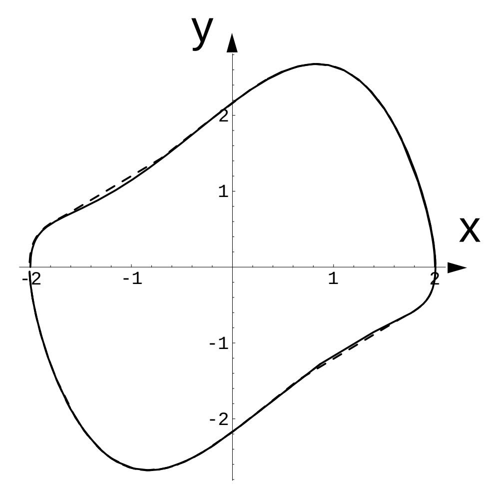
-2
-1
1
2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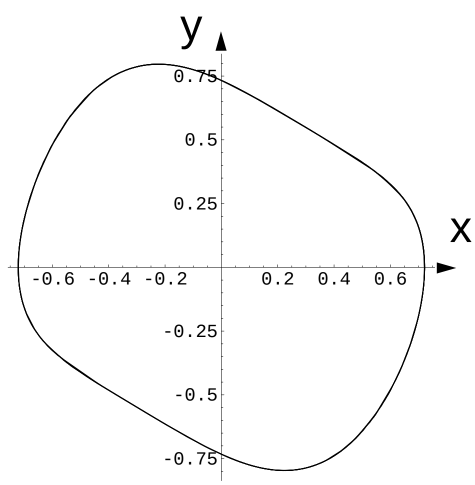
1
2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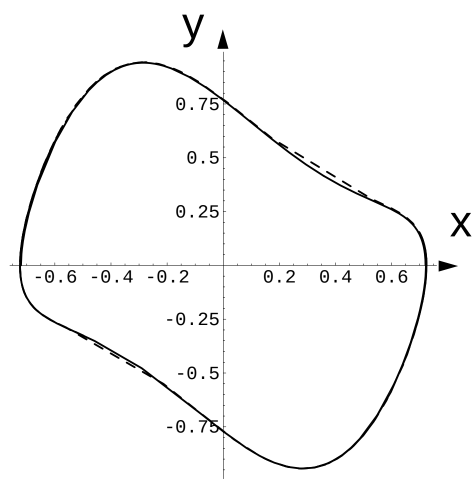
-0.5
-0.75
-0.6 -0.4 -0.2
0.2 0.4 0.6



y
x
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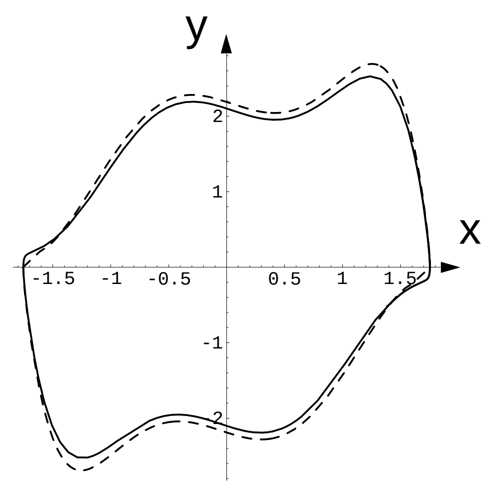
0.25
-0.6 -0.4 -0.2
0.2 0.4 0.6
-0.25
-0.5
-0.75



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-2
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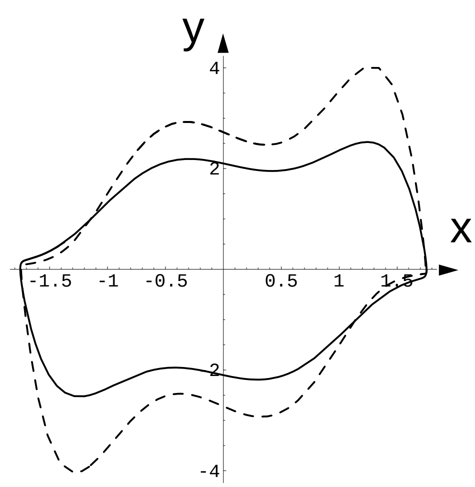
-1
-0.5
0.5
1
1.5



y
x
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2
-1.5
-1
-0.5
0.5
1
1.5
2
-4